LTCC COURSE ON FINITE CLASSICAL GROUPS: EXAM 2025

(1) Let V be a vector space of dimension $n > 1$ over a perfect field of characteristic 2. Let β be a non-degenerate symmetric bilinear form. Let U be the set of isotropic vectors.
   (a) Prove that U is a vector subspace of V.
   (b) Prove that $\dim(U) = n - 1$ if and only if β is not alternating.
   (c) Prove that if β is not alternating, then $\mathrm{Isom}(\beta)$ is a reducible subgroup of $\mathrm{GL}(V)$.

(2) Let V be a vector space of dimension $2r$ over $\mathbb{F}_q$. Let W be a vector subspace of V of dimension m. Let $G = \mathrm{Sp}_{2r}(q)$, the isometry group of a non-degenerate alternating form on V, and let G_W be the stabilizer of W in G.
   (a) Show that if G_W is maximal in G, then $G_W = G_U$ where U is a vector subspace of V that is either non-degenerate or totally isotropic.
   (b) Assuming W is non-degenerate, describe $W^\perp$ and describe G_W.
   (c) Assuming W is totally isotropic, describe $W^\perp$ and describe G_W. A complete description in this case is hard so you may choose to restrict your attention to the case $r = 2$. A description of G_W when $\dim(W) = 1$ was given in class.

(3) Let $n = 2r$ for some positive integer r and let V be an n-dimensional vector space over $\mathbb{F}_{q^2}$. Let $\beta : V \times V \to \mathbb{F}_{q^2}$ be a hermitian σ-sesquilinear form and suppose that we can write

$$V = H_1 \perp H_2 \perp \cdots \perp H_r$$

where $H_1, \ldots, H_r$ are hyperbolic lines with $H_i = \langle e_i, f_i \rangle$ for (e_i, f_i) a hyperbolic pair. Let ζ be an element of $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$ satisfying $\zeta^q = -q$ and define

$$V_1 := \{\lambda_1\zeta e_1 + \lambda_2\zeta e_2 + \cdots + \lambda_r\zeta e_r + \mu_1 f_1 + \mu_2 f_2 + \cdots + \mu_r f_r \mid \lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_r \in \mathbb{F}_q\}.$$

   (a) Prove that V_1 is a vector space over $\mathbb{F}_q$.
   (b) Write $\beta|_{V_1}$ for the restriction of β to $V_1 \times V_1$. Prove that $\beta|_{V_1} = \zeta\beta_1$ where $\beta_1 : V_1 \times V_1 \to \mathbb{F}_q$ is an alternating sesquilinear form.
   (c) Deduce that $\mathrm{Sp}_n(q) \leq \mathrm{SU}_n(q)$ where $\mathrm{SU}_n(q)$ is the set of determinant 1 isometries of a non-degenerate Hermitian form β over a vector space V of dimension n over the field $\mathbb{F}_{q^2}$.

(4) *For this question you may want to refer to the notes for more details about types of quadratic forms.* Let Q be a quadratic form on V, a 2-dimensional vector space over $\mathbb{F}_q$ with q an odd prime power. We say Q is of type O_2^+ if V is a hyperbolic line with respect to Q; we say Q is of type O_2^- if V is anisotropic with respect to Q (there are no singular vectors).

   Write $\mathrm{O}_2^+(q)$ (resp. $\mathrm{O}_2^-(q)$) for the set of isometries of Q in each case.

$$\text{Facts:} \quad |\mathrm{O}_2^+(q)| = 2(q-1) \text{ and } |\mathrm{O}_2^-(q)| = 2(q+1).$$

Use these facts to show that $\mathrm{O}_2^+(q)$ and $\mathrm{O}_2^-(q)$ are both dihedral.