LTCC Applications of Differential Geometry to Mathematical Physics: Exam 2025

(Note: The Einstein summation convention is assumed in question 2.)

1. The complex projective plane $\mathbb{CP}^2 = \mathbb{P}^2(\mathbb{C})$ is the space of complex lines through the origin in $\mathbb{C}^3$. Points in $\mathbb{CP}^2$ are described by projective coordinates $(X_0 : X_1 : X_2)$, i.e. triples of complex numbers (not all zero) defined up to rescaling, so that $(X_0 : X_1 : X_2) = (\lambda X_0 : \lambda X_1 : \lambda X_2)$ for all $\lambda \in \mathbb{C}^*$. If we define

$$U_j = \{\, (X_0 : X_1 : X_2) \,|\, X_j \neq 0 \,\}, \qquad j = 0, 1, 2,$$

then the projective plane can be written as the union

$$\mathbb{CP}^2 = U_0 \cup U_1 \cup U_2.$$

By considering the crossover maps on $U_i \cap U_j$, show that $\mathbb{CP}^2$ is a two-dimensional complex manifold. (Hint: On U_0, take the coordinates (affine coordinates) $z_0 = X_1/X_0$, $w_0 = X_2/X_0$, and similarly for U_1 and U_2.)

2. The standard embedding

$$\iota : \qquad S^2 \longrightarrow \mathbb{R}^3$$

of the 2-sphere in three-dimensional Euclidean space can be presented in the form

$$\iota : \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix},$$

where the angular coordinates are taken in the ranges $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$. (Note: strictly speaking, more than one coordinate chart is required.)

(i) Starting from the standard Euclidean metric g_E on $\mathbb{R}^3$, given by

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2,$$

calculate the induced metric $g = \iota^*(g_E)$ on S^2, and hence show that in the coordinates $(x^1, x^2) = (\theta, \varphi)$ the explicit form of the geodesic Lagrangian $L = \frac{1}{2} g_{jk}\dot{x}^j\dot{x}^k$ is

$$L = \frac{1}{2}\dot{\theta}^2 + \frac{1}{2}\sin^2\theta\,\dot{\varphi}^2. \tag{1}$$

(ii) By calculating the Euler-Lagrange equations for (1), write down the geodesic equations.

(iii) Use a Legendre transformation to find the canonical conjugate momenta $(p_1, p_2) = (p_\theta, p_\varphi)$ associated with (θ, φ), and the Hamiltonian H for the geodesic flow on S^2.

(iv) Show that this is a Liouville integrable system, by verifying that $K = p_\varphi$ is a second independent constant of motion, and hence (for fixed $H = \text{const}$, $K = \text{const}$) reduce the equations of motion to the quadrature

$$\pm \int \frac{\sin\theta\,\mathrm{d}\theta}{\sqrt{2H\sin^2\theta - K^2}} = t + \text{const}$$

for $\theta(t)$, together with another the quadrature for $\varphi(t)$, which you should find. Can you describe the shape of the geodesics obtained when $K = 0$?

3. In one spatial dimension, the differential operator

$$\mathcal{J}_1 = D_x$$

(total derivative with respect to x) is Hamiltonian, meaning that it defines a Poisson bracket on pairs of functionals F, G of a field u according to

$$\{F, G\} = \left\langle \frac{\delta F}{\delta u}, \mathcal{J}_1 \frac{\delta G}{\delta u} \right\rangle,$$

where $\delta/\delta u$ denotes the variational derivative (as in lectures). The 3rd order differential operator

$$\mathcal{J}_2 = D_x^3 + 4uD_x + 2u_x$$

is also Hamiltonian, and is compatible with $\mathcal{J}_1$ in the sense that the sum $\mathcal{J}_1 + \mathcal{J}_2$ (or any linear combination of them) also defines a Poisson bracket on functionals of the field u.
(i) Verify that the 5th order KdV equation

$$u_t = u_{xxxxx} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x \tag{2}$$

(with subscripts denoting partial derivatives) can be written in the Hamiltonian form

$$u_t = \mathcal{J}_1 \frac{\delta H_1}{\delta u}, \quad \text{where} \quad H_1 = \int_{-\infty}^{\infty} \left(\tfrac{1}{2}u_{xx}^2 - 5uu_x^2 + \tfrac{5}{2}u^4 \right) \, \mathrm{d}x.$$

(Hint: for the variational derivative, apply the Euler operator to the density for H_1.)
(ii) By making suitable assumptions about the behaviour of u and its derivatives as $|x| \to \infty$, show that the Hamiltonian H_1 is a conserved quantity for the 5th order PDE (2).
(iii) By calculating the vector field

$$\mathcal{J}_2 \frac{\delta H_2}{\delta u},$$

with the functional

$$H_2 = \int_{-\infty}^{\infty} \left(-\tfrac{1}{2}u_x^2 + u^3 \right) \, \mathrm{d}x,$$

show that the 5th order KdV equation (2) is bi-Hamiltonian.

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