LTCC COURSE ON FINITE CLASSICAL GROUPS: MOCK EXAM 2025

Instructions: These exercises are designed to give you an idea of what you can expect in the exam. The exam will have a similar number of questions.

The questions on this sheet broadly focus on two themes: (a) isomorphisms between classical groups and other groups; (b) subgroups of classical groups.

(1) Let V be a vector space of dimension n over $\mathbb{F}_q$. Let W be a vector subspace of V of dimension m. Describe the stabilizer of W in the group $\mathrm{GL}_n(q)$.

(2) Identify the field of order 9 with the set

$$\{a + bi \mid a, b \in \mathbb{F}_3\}$$

where $i^2 = -1$. Let

$$X = \begin{pmatrix} 1 & 1+i \\ 0 & 1 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(a) Calculate the order of X, Y and XY.

(b) Use the fact that

$$\langle a, b \mid a^5 = b^3 = (ab)^2 = 1 \rangle \cong A_5$$

to prove that $A_6 \cong \mathrm{PSL}_2(9)$.

(3) Let V be the space of 4×4 skew-symmetric matrices over the field $\mathbb{F}_q$ where q is odd.

(a) Show that V is a 6-dimensional vector space;

(b) Show that $\mathrm{GL}_4(q)$ acts linearly on V by $X^A = A^T X A$.

Define a function

$$Q : V \to \mathbb{F}_q,\ X \mapsto X_{12}X_{34} - X_{13}X_{24} + X_{14}X_{23}.$$

(c) Show that Q is a non-degenerate quadratic form.

(d) Show that if A is a diagonal or elementary matrix, then $Q(X^A) = \det(A)Q(X)$. (You do not need to show all working for the elementary matrices – just describe your strategy).

(e) Deduce that $\mathrm{SL}_4(q)/\langle -I \rangle$ is a subgroup of the isometry group of Q.

Remark: Q is known as the *Pfaffian* of the matrix X. It turns out that Q is of $+$-type, so we find that $\mathrm{SL}_4(q)/\langle -I \rangle \leq \mathrm{Isom}(Q) = \mathrm{O}_6^+(q)$. Comparing orders, one sees that $\mathrm{SL}_4(q)/\langle -I \rangle$ is actually an index 4 subgroup of $\mathrm{Isom}(Q)$.

(4) Show how to obtain the following embeddings:

(a) $\mathrm{GL}_n(q^m) < \mathrm{GL}_{mn}(q)$;

(b) $\mathrm{GL}_n(q) < \mathrm{GL}_n(q^m)$;

(c) $\mathrm{GL}_n(q) \wr \mathrm{Sym}(m) < \mathrm{GL}_{mn}(q)$.

In every case, q is a power of a prime; m and n are positive integers. The symbol "$\wr$" denotes a wreath product.