London Taught Course Centre

2024/25 examination

Graph Theory

Instructions to candidates

This open-book exam has 2 questions. Some parts are harder than others. Substantial credit will be given for partial answers and ideas which you cannot justify, provided that you clearly distinguish between statements which you believe but do not see how to prove, statements which you believe you have proved, and statements you think are obvious enough not to need a proof.

You may wish to use Internet searches in addition to the lecture notes. This is allowed and encouraged. You are also allowed to use any theorems you find, provided they are properly referenced.

1 Given a graph G, the *square* of G, written G^2, is the graph on $V(G)$ with edges $xy \in E(G^2)$ if and only if there is a path with at most two edges in G from x to y. Recall that $\Delta(G)$ is the maximum degree of G, and $\chi(G)$ is the chromatic number of G.

(a) Show that if T is a tree, then $\chi\big(T^2\big) = \Delta(T) + 1$.

(b) Show that if G is a planar graph, and $\Delta(G) \geq 5$, then $\chi(G^2) \leq 9\Delta(G) - 19$.

(c) For which $\Delta = \Delta(G)$ is it true that if G is a planar graph, then $\chi\big(G^2\big) \leq \frac{3}{2}\Delta(G) + 5$?

2 (a) Let Σ be the alphabet $\{\#, 0, 1, +\}$. We say that a word $s \in \Sigma^*$ is an "addition problem" if s is of the form "$x + y$" where x and y are words in $\{0,1\}^*$ such that $|x| \geq |y| > 0$. Define a state set and transition function of a Turing machine T that halts in polynomial time and satisfies the following: if the input on the tape is an addition problem, then after the machine halts, the non-negative half of the tape should have the form $z\#w$ where z is the sum of x and y as integers in binary representation (most significant digits are on the left) and w is an arbitrary word. If the input is not an addition problem, the content of the tape should remain unchanged. All binary representations of integers have their most significant digit on the left.

(b) Show that for each $\gamma > 0$ there is a polynomial-time algorithm which takes as input a graph G on n vertices and which has the following behaviour. If G has a proper 3-colouring, then the algorithm must return 'Yes'. If, for every $S \subseteq E(G)$ with $|S| \leq \gamma n^2$, the graph $G - S$ does not have a proper 3-colouring, then the algorithm must return 'No'. If neither condition is satisfied, the algorithm may answer either 'Yes' or 'No'. *You may use the fact that there is an algorithm which takes as input a graph G and a parameter $\varepsilon > 0$, which returns an ε-regular partition of $V(G)$ with between ε^{-1} and K parts, and whose running time, for any fixed $\varepsilon > 0$, is polynomial in n. Here $K = K(\varepsilon)$ does not depend on n.*
You may describe your algorithm in words, give pseudocode, or implement it with a Turing machine, as you prefer.