LTCC: Pseudodifferential operators and applications to PDEs

Examiners: Claudia Garetto

 Turn Over

Question 1 [45 marks].

(i) Give the definition of the symbol class $S^m(\mathbb{R}^{2n})$.
Prove that if $a \in S^{m_1}(\mathbb{R}^{2n})$ and $b \in S^{m_2}(\mathbb{R}^{2n})$ then $\partial_\xi^\alpha a \partial_x^\beta b \in S^{m_1+m_2-|\alpha|}(\mathbb{R}^{2n})$, for all $\alpha, \beta \in \mathbb{N}_0^n$. **[10]**

(ii) Give the definition of pseudodifferential operator $a(x, D)$ with symbol $a \in S^m(\mathbb{R}^{2n})$. Write $a(x, D)f$, $f \in \mathscr{S}(\mathbb{R}^n)$ in the oscillatory integral form

$$\int_{\mathbb{R}^{2n}} \mathrm{e}^{-iy\xi} \left(\mathrm{e}^{ix\xi} a(x, \xi) f(y) \right) dy \, đ\xi$$

and prove that

$$b_x(y, \xi) = \mathrm{e}^{ix\xi} a(x, \xi) f(y)$$

is an amplitude of order $m_+ := \max\{0, m\}$ with respect to y and ξ. **[15]**

(iii) Let $x_0 \in \mathbb{R}^n$. Define the translation operator

$$T_{x_0} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n) : f \to f(\cdot + x_0).$$

(a) Prove that T_{x_0} can be written in the integral form

$$T_{x_0}(f)(x) = \int_{\mathbb{R}^n} \mathrm{e}^{ix\xi} \mathrm{e}^{ix_0\xi} \widehat{f}(\xi) \, đ\xi.$$ **[10]**

(b) Is $a(\xi) = \mathrm{e}^{ix_0\xi}$ a symbol in $S^m(\mathbb{R}^{2n})$? **[10]**

Question 2 [55 marks].

(i) Define the Sobolev space $H^s(\mathbb{R}^n)$ with $s \in \mathbb{R}$. Using (without proof) the fact that pseudodifferential operators of order 0 are bounded on $L^2(\mathbb{R}^n) = H^0(\mathbb{R}^n)$ state and prove the theorem of Sobolev boundedness for $a(x, D)$ with $a \in S^m(\mathbb{R}^n)$, $m \neq 0$. **[10]**

(ii) Let $a \in S^m(\mathbb{R}^{2n})$.

(a) Give the definition of (pseudodifferential) right parametrix of $a(x, D)$. **[5]**

(b) Prove that if $a(x, D)$ has a right parametrix $p(x, D)$ then it is a hypoelliptic operator, i.e., $\operatorname{sing\,supp} a(x, D)u = \operatorname{sing\,supp} u$ for all $u \in \mathscr{S}'(\mathbb{R}^n)$. **[10]**

(iii) Let $m \in \mathbb{R}$ and $l \leq m$.

(a) Define the class of hypoelliptic symbols of type (m, l). **[5]**

(b) Define the class of elliptic symbols of order l and prove that an elliptic symbol of order m is hypoelliptic. **[10]**

(iv) Let $a \in S^m(\mathbb{R}^{2n})$.

(a) Prove that if there exists $p \in S^{-m}(\mathbb{R}^{2n})$ such that $p\sharp a - 1 = r$, where $r \in S^{-\infty}(\mathbb{R}^{2n})$ then a is an elliptic symbol of order m. **[10]**

(b) If a is a hypoelliptic symbol of type (m, l) and u solves the equation $a(x, D)u = f$, where $f \in \mathscr{S}(\mathbb{R}^n)$, can u be non smooth? Justify your answer. **[5]**