Analytical Methods Exam 2024.

1. Determine the first two terms of the asymptotic expansion for the solution $u(x, y)$ to the following partial differential equation:

$$\frac{1}{4}\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \varepsilon x^2 u = -1,$$

where $\varepsilon \ll 1$ is a small positive parameter. Your asymptotic expansion should satisfy the following boundary conditions:

$$\frac{\partial u}{\partial y}(x, 0) = 0 \quad \text{and} \quad u(x, 0) = 0 \qquad \text{for } -\infty < x < \infty.$$

Hint: start by using the general solution to the inhomogeneous wave equation as given in the notes. For what part of the xy-plane is this expansion valid?

2. Using the method of steepest descents, determine the asymptotic behaviour of

$$\psi(x) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} \frac{\exp x(2s - \sqrt{s})}{s} \, \mathrm{d}s,$$

as $x \to \infty$, where $\gamma > 0$ is a real positive parameter (so that the line of integration lies to the right of the origin).