You may refer without proof to results from the course (theorems, examples, etc).

Q1 Let (X, Y) be a two-dimensional random variable with (bivariate) distribution function

$$F(x, y) = \mathbb{P}[X \le x, Y \le y], \quad (x, y) \in \mathbb{R}^2.$$

(i) For $x_1 < x_2, y_1 < y_2$ express explicitly in terms of F the probability measure of rectangle $(x_1, x_2] \times (y_1, y_2]$ given by

$$\mu_F((x_1, x_2] \times (y_1, y_2]) := \mathbb{P}[(X, Y) \in (x_1, x_2] \times (y_1, y_2]].$$

(ii) For $G = G(x, y)$ suppose $\mu_G((x_1, x_2] \times (y_1, y_2])$ is defined in terms of G as in part (i). Suppose that $\mu_G((x_1, x_2] \times (y_1, y_2]) \ge 0$ for all rectangles and that $\lim_{x,y\to\infty} G(x, y) = 1$. Sketch an argument showing that there exists a unique probability measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ that coincides with $\mu_G((x_1, x_2] \times (y_1, y_2])$ for all rectangles $(x_1, x_2] \times (y_1, y_2]$.

(iii) Show that F is continuous if and only if the marginal distribution functions $F_X(x) = \mathbb{P}[X \le x]$ and $F_Y(y) = \mathbb{P}[Y \le y]$ are continuous.

(iv) Is it possible that $\mathbb{P}[X = x, Y = y] = 0$ for all $x, y \in \mathbb{R}$ but F is not continuous? Either prove the impossibility of this claim or give an example.

(v) For $F(x, y) = \min(x, y)$, where $(x, y) \in [0, 1] \times [0, 1]$, determine the marginal distributions of X and Y. Are X and Y independent?

(vi) Give an example of F such that (X, Y) has a (joint) density, X is uniformly distributed on $[0, 1]$, Y is uniformly distributed on $[0, 1]$, and the random variables are not independent.

Q2 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $T : \Omega \to \Omega$ be a measurable map preserving the probability measure $\mathbb{P}$, that is satisfying $\mathbb{P}[T^{-1}(A)] = \mathbb{P}[A]$ for $A \in \mathcal{F}$. For a given (real-valued) random variable X_0, define a sequence $X_0, X_1, X_2, \dots$ recursively as

$$X_n(\omega) = X_{n-1}(T(\omega)), \text{for } \omega \in \Omega, n \in \mathbb{N}. \tag{1}$$

(i) Show that the sequence $X_0, X_1, \dots$ is *stationary*, that is satisfies $(X_0, \dots, X_n) \stackrel{d}{=} (X_k, \dots, X_{k+n})$ for all $k, n \in \mathbb{N}$ (where $\stackrel{d}{=}$ denotes the identity in distribution).

(ii) Give an example (construction) of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a measure-preserving T and a random variable X_0 such that $X_0, X_1, \dots$ in (1) are i.i.d. (independent, identically distributed) with given distribution function $F(x) = \mathbb{P}[X_0 \leq x]$. [Hint: $\Omega = \mathbb{R}^\infty$.]

(iii) A set $A \in \mathcal{F}$ is said to be *invariant* if $T^{-1}(A) = A$. The collection of all invariant sets is called the invariant σ-algebra $\mathcal{I}$. Show that $\mathcal{I}$ is indeed a σ-algebra.

(iv) A random variable X is said to be *invariant* if $X(\omega) = X(T(\omega))$. Show that such invariant X are precisely the $\mathcal{I}$-measurable random variables.

(v) We call T *ergodic* if $\mathbb{P}[A] = 0$ or $\mathbb{P}[A] = 1$ for every $A \in \mathcal{I}$. Show that T is ergodic if and only if every invariant random variable X is constant with probability one.

(vi) For the probability space in part (ii), show that $\mathcal{I} \subset \mathcal{T}$, where $\mathcal{T}$ is the tail σ-algebra generated by $X_0, X_1, \dots$ Use this connection to conclude that T in part (ii) is ergodic.

(vii) Consider the setting where $\Omega = [0, 1)$, $\mathcal{F}$ is the Borel σ-algebra, $\mathbb{P}$ is the Lebesgue measure and for given $\alpha \in \mathbb{R}$

$$T(\omega) = \{\omega + \alpha\},$$

where $\{x\}$ denotes the fractional part of x.

(1) For

$$\alpha = 1/3,$$

construct explicitly an invariant set of measure $2/3$.

(2) For

$$\alpha \in \mathbb{Q},$$

show that T is not ergodic.

(3) For α irrational number, prove that T is ergodic. [Hint: it is known that every measurable function on $[0, 1)$ with $\int_0^1 f^2(\omega) \mathrm{d}\omega < \infty$ has a unique Fourier series representation

$$\sum_{k \in \mathbb{Z}} c_k e^{2\pi \mathtt{i} k \omega}.$$

Use this fact to determine all invariant indicator functions $f(\omega) = 1(\omega \in A)$.]

Q3 Let $(B(t),\ t \geq 0)$ be a Brownian motion (BM), with the natural filtration $(\mathcal{F}_t, t \geq 0)$, where $\mathcal{F}_t = \sigma(B(s), s \in [0,t])$. Consider for $t \geq 0$

$$I(t) := \int_0^t s^{-1} B(s) \mathrm{d}s, \quad W(t) := B(t) - I(t).$$

(i) The integral involved is an improper Riemann integral, defined as

$$I(t) = \lim_{\varepsilon \to 0+} \int_\varepsilon^t s^{-1} B(s) \mathrm{d}s.$$

Show that the limit indeed exists almost surely. [Hint: you may use Hölder continuity of the BM and the dominated convergence to show this.]

(ii) Show that the process $(W(t),\ t \geq 0)$ is adapted to the filtration $(\mathcal{F}_t, t \geq 0)$.

(iii) Show that the process $(W(t),\ t \geq 0)$ is Gaussian. [Hint: for $0 < t_1 < \cdots < t_n$ approximate $I(t_1), \ldots, I(t_n)$ by Riemann sums, argue that they are jointly normal, and use the fact that a weak limit of multivariate normal vectors also has a multivariate normal distribution.]

(iv) Determine explicitly the joint distribution of the random variables $B(1)$ and $I(2)$.

(v) Determine $\mathbb{E}[I(2)|B(1)]$ and $\mathbb{E}[I(2)|\mathcal{F}_1]$.

(vi) Determine if $(W(t), t \geq 0)$ is a martingale with respect to the filtration $(\mathcal{F}_t, t \geq 0)$. [Hint: you need to prove or disprove that $\mathbb{P}[W(t)|\mathcal{F}_s] = W(s)$ holds for $0 \leq s < t$.]

(vii) Find $\operatorname{cov}(B(s), I(t))$ and $\operatorname{cov}(I(s), I(t))$ for $s, t \geq 0$.

(viii) Prove that $(W(t), t \geq 0)$ is a BM [Hint: use the Gaussian characterisation of BM.]