MTP2024

Q1 (i) The function F is nondecreasing, right-continuous with left limits. For instance,

$$(-\infty, x) = \bigcup_n (-\infty, x - 1/n)$$

and $(-\infty, x-1/n) \subset (-\infty, x - 1/{n+1})$ imply left continuity.

If $F(x-) < F(x)$, then $\mu(\{x\}) > 0$ and μ has atom at x.

Thus F is continuous, hence takes all intermediate values

$$F(-\infty) = 0 < t < \mu(\mathbb{R}) = F(\infty).$$

(ii) (1). For $B \subset A$, $\mu(B) \le \frac{1}{2}\mu(A)$ or $\mu(A \setminus B) \le \frac{1}{2}\mu(A)$,

so we can find $B_1 \subset A$ s.t. $0 < \mu(B_1) \le \frac{1}{2}\mu(A)$.

Since B_1 is not atom, there exists $B_2 \subset B_1$ s.t.

$$0 < \mu(B_2) \le \frac{1}{2}\mu(B_1) \le \frac{1}{4}\mu(A).$$

Continuing so forth, construct $B_1 \supset B_2 \supset B_3 \supset \dots$

with $\mu(B_k) \le \frac{1}{2^k}\mu(A)$.

Now choose suitable k.

(2) By (1) we have $0 < \varphi(A) \le \varepsilon$ if $\mu(A) > 0$ and

there exists A_1 with $0 < \mu(A_1) \le \varphi(\Omega)$.

Note also that $A \subset B$ implies $\varphi(A) \le \varphi(B)$.

If $\mu(\Omega \setminus A_1) = 0$ we are done, otherwise $\varphi(\Omega \setminus A_1) > 0$ and by the definition of φ there exists $A_2 \subset \Omega \setminus A_1$ such that

$$\frac{1}{2}\varphi(\Omega \setminus A_1) \le \mu(A_2) \le \varphi(\Omega \setminus A_1) \le \varepsilon .$$

Continuing so forth, either we construct a finite sequence $A_1, \dots, A_n$ with $\mu(\Omega \setminus \bigcup_{k=1}^{n} A_k) = 0$ or infinite sequence of disjoint $A_1, A_2, \dots$. In the latter case for $A_0 := \Omega \setminus \bigcup_{k=1}^{\infty} A_k$ we get

$$\varphi(A_0) \le \varphi\Big(\Omega \setminus \bigcup_{k=1}^{n} A_k\Big) \le 2\mu(A_{n+1}) \qquad n = 1, 2, \dots$$

But $\sum_n \mu(A_n) \le \mu(\Omega) < \infty$ hence $\mu(A_n) \to 0$ and $\varphi(A_0) = 0$ implies $\mu(A_0) = 0$. Finally, choose n to have $\sum_{k=n}^{\infty} \mu(A_k) < \varepsilon$; then

$$A_1, \dots, A_{n-1}, \Big(\bigcup_{k=n}^{\infty} A_k\Big) \cup A_0$$

is a partition of Ω with desired property.

(iii) Using (ii) it is possible to construct $A_1 \subset A_2 \subset \dots$ with $\mu(A_n) \to t$. Choose k large enough and find $B_1, \dots, B_m$ disjoint with $\mu(B_i) \leq 1/k$ and $\bigcup_{i=1}^{m} B_i = \Omega$.

Take a subset $B_1, \dots, B_\ell$ such that

$$\mu(B_1 \cup \dots \cup B_\ell) \leq t < \mu(B_1 \cup \dots \cup B_\ell \cup B_{\ell+1})$$

then $\quad t - \mu(B_1 \cup \dots \cup B_\ell) \leq \frac{1}{k}$.

Then partition $\Omega \setminus (B_1 \cup \dots \cup B_\ell)$ to find a better approximation to t, in terms of measure.

For $A_1 \subset A_2 \subset \dots$ sequence of such approximations take $A = \bigcup_n A_n$ to obtain $\mu(A) = t$.

Yes, it is enough to show existence of A with $\mu(A) = \frac{1}{2}\mu(\Omega)$. Binary expansion of t tells how iterates should be constructed.

Q2 (i) $\mathcal{F}_n = \sigma(X_1, \dots, X_n) = \sigma(R_1, \dots, R_n)$, and conditioning on $\mathcal{F}_n$ gives

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[X_{n+1} \mid \sigma(R_1, \dots, R_n)] = \mathbb{E}[X_{n+1} \mid R_n]$$

because $R_1, R_2, \dots$ is a Markov chain. Set $R_n = m$, so $X_n = \frac{m}{nd+r+g}$, then computing transition probabilities for $m \to m+d$ and $m \to m$ gives

$$\mathbb{E}[X_{n+1} \mid R_n = m] = \frac{m+d}{nd+d+r+g} \frac{m}{nd+r+g}$$

$$+ \frac{m}{nd+d+r+g} \frac{nd+r+g-m}{nd+r+g} = \frac{m}{nd+r+g}$$

Thus $\mathbb{E}[X_{n+1} \mid R_n] = \frac{R_n}{nd+r+g}$

$$\Leftrightarrow \quad \mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[X_{n+1} \mid X_n] = X_n .$$

(ii) This follows by martingale convergence, by also noting that $0 \le X_n \le 1$ so (X_n) is uniformly integrable.

(iii) $R_{n+1} = 1$ if $R_n = 1$ and green ball drawn at step $n+1$, so

$$\mathbb{P}(R_{n+1} = 1) = \frac{1}{n+1} \, \frac{n+1}{n+2} = \frac{1}{n+2},$$

where we assumed by induction that R_n is uniform.

Similarly, for $1 < m \leq n+1$

$$\mathbb{P}(R_{n+1} = m) = \frac{1}{n+1} \, \frac{m-1}{n+2} + \frac{1}{n+1} \, \frac{n+2-m}{n+2} = \frac{1}{n+2}.$$

This proves induction step.

(iv) Appropriate filtration is $\mathcal{G}_n = \sigma(R_n, R_{n+1}, \dots)$,

so $E\left[\frac{R_n - 1}{n} \,\middle|\, \mathcal{G}_{n+1}\right] = E\left[\frac{R_n - 1}{n} \,\middle|\, R_{n+1}\right]$ by

Markov property. The backward transition probability is computed as (using uniform distribution)

$$\mathbb{P}(R_n = m-1 \mid R_{n+1} = m) = \frac{\mathbb{P}(R_n = m-1, R_{n+1} = m)}{\mathbb{P}(R_{n+1} = m)} =$$

$$= \frac{\mathbb{P}(R_{n+1} = m \mid R_n = m-1)\, \mathbb{P}(R_n = m-1)}{\mathbb{P}(R_{n+1} = m)} =$$

$$\frac{\frac{m-1}{n+2} \quad \frac{1}{n+1}}{\frac{1}{n+2}} = \frac{m-1}{n+1}$$ whence

$$\mathbb{P}(R_n = m \mid R_{n+1} = m) = \frac{n+2-m}{n+1}.$$

The conditional expectation becomes

$$E\left[\frac{R_n - 1}{n} \,\Big|\, R_{n+1} = m\right] = \frac{m-1}{n+1}\,\frac{m-2}{n} + \frac{n+2-m}{n+1}\,\frac{m-1}{n}$$

$$= \frac{m-1}{n+1}$$

That is $\mathbb{E}\left[\frac{R_n - 1}{n} \,\Big|\, \mathcal{G}_{n+1}\right] = \mathbb{E}\left[\frac{R_n - 1}{n} \,\Big|\, R_{n+1}\right] =$

$$\frac{R_{n+1} - 1}{n+1}$$

$\Rightarrow \left(\frac{R_n - 1}{n},\ n = 1, 2, \ldots\right)$ is a reversed martingale.

Q3 (i) Suppose $s < u < t$, then by the Markov property

$$\mathbb{E}\left[B(t) \mid B(s) = x, B(u) = y\right] = E\left[B(t) \mid B(u) = y\right] = y$$

because the BM is a martingale. Case $t < s < u$ is similar.

Suppose $t < s < u$, then by independence of increments

$$\mathbb{E}\left[B(t) \mid B(s) = x, B(u) = y\right] = \mathbb{E}\left[B(t) \mid B(s) = x\right] =$$

$$= \mathbb{E}\left[\left(B(t) - \tfrac{t}{s} B(s)\right) + \tfrac{t}{s} B(s) \,\Big|\, B(s) = x\right] =$$

$$= \mathbb{E}\left[B(t) - \tfrac{t}{s} B(s)\right] + \tfrac{t}{s}\,\mathbb{E}\left[B(s) \mid B(s) = x\right] = \tfrac{t}{s}\, x$$

where we used that $B(t) - \frac{t}{s} B(s)$ and $B(s)$ for $t<s$ are independent. Indeed, they are jointly normal and

$$\operatorname{Cov}\left(B(t) - \tfrac{t}{s} B(s), B(s)\right) = t - \frac{t}{s} \cdot s = 0.$$

Case $t<u<s$ is similar.

The answer in the case $\cancel{t<u<s}$ $s<t<u$ is

$$\mathbb{E}\left[B(t) \mid B(s) = x, B(u) = y\right] = \frac{t-s}{u-s}\, x + \frac{u-t}{u-s}\, y.$$

This is reduced to the case $\mathbb{E}[B(s) \mid B(t)]$, $s<t$ using that for fixed $c>0$

$W(t) = B(t+c) - B(c)$ is a BM independent of the path $(B(p),\ p \le c)$.

(ii) (a) Let $M(t) = \max\{B(s),\ s \le t\}$.

Then $\{\tau_x \le t\} = \{M(t) \ge x\}$ and (see Lecture 5)

$$\mathbb{P}[M(t) \ge x] = \mathbb{P}\left[M(1) \ge \frac{x}{\sqrt{t}}\right] = \mathbb{P}\left[|B(1)| \ge \frac{x}{\sqrt{t}}\right]$$

$$= \sqrt{\frac{2}{\pi}} \int_{x/\sqrt{t}}^{\infty} e^{-y^2/2}\, dy.$$ Differentiating in t gives the density of τ_x:

$$f(t) = \frac{x}{t^{3/2}\sqrt{2\pi}}\, e^{-\frac{x^2}{2t}}, \quad t>0.$$

Then $\mathbb{E}\tau_x = \int_0^\infty t f(t)\, dt = \infty.$

(b) By the strong Markov property

$W(t) = B(t+\tau_x) - B(\tau_x)$ is a BM independent of

τ_x. Therefore the time for W to reach level y has

same distribution as τ_y. The claim follows since

W reaches y when B reaches $x+y$.

(c) This is shown by induction, similarly to (b).

(iii) Given $\tau_x = t$ the distribution of $W(\tau_x)$ is $N(0,t)$.

Therefore by total probability, the density of $W(\tau_x)$

is

$$g(y) = \int_0^\infty \frac{1}{\sqrt{2\pi t}} e^{-y^2/2t} \frac{x}{t^{3/2}\sqrt{2\pi}} e^{-\frac{x^2}{2t}} dt =$$

substitute $z = \frac{1}{2t}(x^2+y^2)$

$$= \int_0^\infty \frac{x}{\pi(x^2+y^2)} e^{-z} dz = \frac{x}{\pi(x^2+y^2)}$$

Which is the density function (in variable y) of the

Cauchy distribution with parameter x.