Measure Theory:

The λ^* stands for Lebesgue outer measure and λ for the Lebesgue measure defined on Lebesgue measurable sets.

1. Let $f_n = \frac{1}{n^2}\sin(n^3 x)$. Show that $f = \sum_{n=1}^{\infty} f_n$ is a Lebesgue measurable function and $\int_I f(x)dx = \sum_{n=1}^{\infty}\int_I f_n(x)dx$ for every finite interval I.

2. For every $i = 1, 2, \ldots$ and integers n define $A_{n,i} := \{x \mid \frac{n}{i} \leq x < \frac{n+1}{i}\}$.

(a) What is the smallest algebra containing all the $A_{n,i}$?

(b) What is the smallest sigma algebra containing all the $A_{n,i}$?

(c) Given an example of two different algebras $\mathcal{A}$ and $\mathcal{B}$ such that the generated sigma algebras $\sigma(\mathcal{A})$ and $\sigma(\mathcal{B})$ are the same.

3. Let A be a subset of the real numbers and r be a real number with $0 < r < 1$ such that for every integer k and positive integer n it follows that

$$\frac{r}{2^n} \leq \lambda^*(A \cap [\frac{k}{2^n}, \frac{k+1}{2^n}]) \leq \frac{1-r}{2^n}.$$

Show that A is not a Lebesgue measurable set.

4 Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous and non-decreasing function. Show that $g(x) = \liminf_{n\to\infty} n\ (f(x+\frac{1}{n}) - f(x))$ is a measurable function and for every $a < b$ it follows that $\int_a^b g(x)dx \leq f(b) - f(a)$. Show that there is such a function f with g so defined as above such that $\int_a^b g(x)dx < f(b) - f(a)$.

5.a. Show that for every Lebesgue measurable subset A of the real numbers there is a Borel subset B such that $A \subseteq B$ and $\lambda(B \backslash A) = 0$.

5.b. Show that there is a Lebesgue measurable subset A of the real numbers such that there is no closed subset C such that $A \subseteq C$ and $\lambda(C \backslash A) = 0$.

6. Let $\mathbf{Z}$ be the integers and let μ be a finitely additive probability measure on all the subsets of $\mathbf{Z}$ such that for every $k = 1, 2, 3, \ldots$ and $j \in \mathbf{N}$ it follows that $\mu(\{kn + j \mid n = \ldots, -2, -1, 0, 1, 2 \ldots\}) = \frac{1}{k}$.

(a) Show that μ cannot be extended to a sigma-additive measure on all the subsets of $\mathbf{Z}$.

(b) What is the smallest algebra $\mathcal{A}$ of subsets including the above sets $\{kn + j \mid n = \ldots, -2, -1, 0, 1, 2 \ldots\}$ and for this $\mathcal{A}$ determine $\sigma(\mathcal{A})$.

7. Let $f_1, f_2, \ldots, f_k$ and $g_1, g_2, \ldots, g_k$ be functions from the real numbers to the real numbers such that each of the f_i and each of the g_i are either non-decreasing or non-increasing. Show that the function $\sum_{i=1}^{k} g_i f_i$ is Borel measurable.

8. Let A and B be subsets such that such that $A \cap B = \emptyset$, $\mathbf{R} = A \cup B$, with $A + A = \{a_1 + a_2 \mid a_1, a_2 \in A\} \subseteq A$, $B + B = \{b_1 + b_2 \mid b_1, b_2 \in B\} \subseteq B$ and that both A and B contain both at least one negative and positive number. Show that A and B cannot be Lebesgue measurable.