LTCC exam: Pseudo-differential operators

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Question 1.

1. Prove that if $f \in L^1(\mathbb{R}^n)$ then $\widehat{f}$ is continuous everywhere.

2. Let $u, f \in L^1(\mathbb{R}^n)$ be such that $f = \Delta_x u$, where $\Delta_x := \sum_{k=1}^n \partial_{x_i}^2$ is the Laplacian. Show that
$$\int_{\mathbb{R}^n} f(x)dx = 0.$$

3. Let $u, f \in L^1(\mathbb{R}^n)$ satisfy $(1 - \Delta_x)u = f$. Suppose that f satisfies
$$|\widehat{f}(\xi)| \leq \frac{C}{(1+|\xi|)^{n-1}}.$$
Prove that u is a bounded continuous function on $\mathbb{R}^n$.

4. Let $f \in L^1(\mathbb{R}^n)$ and let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix. If $B = (A^t)^{-1}$ then
$$\widehat{f \circ A} = |\det(A)|^{-1} \widehat{f} \circ B.$$

5. Prove that $\widehat{1} = \delta$.

Question 2.

1. Prove the following properties of the classes S^m, $m \in \mathbb{R}$.
   A. If $\sigma \in S^{m_1}$ and $\tau \in S^{m_2}$ then $\sigma\tau \in S^{m_1+m_2}$.
   B. If $\sigma \in S^m$, then $\partial_x^\beta \partial_\xi^\alpha \sigma \in S^{m-|\alpha|}$ for all $\alpha, \beta \in \mathbb{N}_0$.
   C. Let $\sigma \in S^m$ and let $\phi \in \mathcal{S}(\mathbb{R}^n)$. Show that $\tau(x,\xi) := \sigma(x,\xi)\phi(\xi)$ defines a symbol in $S^{-\infty} := \bigcap_{m\in\mathbb{R}} S^m$.

2. Let $P(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x)\partial_x^\alpha$, be a partial differential operator of order $m \in \mathbb{N}$ with coefficients $a_\alpha \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Show that $P(x,D)$ has a symbol $(x,\xi) \mapsto P(x,\xi) \in S^m$.

3. Show that the pseudo-differential operator with symbol $\sigma(x,\xi) = e^{-\frac{|\xi|^2}{2}}$ does not map $C_0^\infty(\mathbb{R}^n)$ into $C_0^\infty(\mathbb{R}^n)$.

4. Let $a \in S^m$ and let $\gamma \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be such that $\gamma = 1$ near the origin. For $\epsilon > 0$ define $a_\epsilon(x,\xi) = a(x,\xi)\gamma(\epsilon x, \epsilon\xi)$. Prove that $a_\epsilon \in S^m$ uniformly in $0 < \epsilon \leq 1$ (i.e. show that the constants in symbolic inequalities may be chosen independent of $0 < \epsilon \leq 1$) and that $\partial_x^\alpha \partial_\xi^\beta a_\epsilon(x,\xi) \to \partial_x^\alpha \partial_\xi^\beta a(x,\xi)$ as $\epsilon \to 0$ for all $x, \xi \in \mathbb{R}^n$.

5. Suppose that $b \in S^m$ and $b \geq 0$ is elliptic of order m. Prove that $a = \sqrt{b} \in S^{\frac{m}{2}}$.