LTCC Examination 2025

Symmetry Methods for Differential Equations

1. (a) Show that the hyperbolic rotation group

$$x^* = x\cosh\varepsilon + ct\sinh\varepsilon, \qquad t^* = \frac{x\sinh\varepsilon}{c} + t\cosh\varepsilon, \tag{1}$$

with c a non-zero constant, forms a one-parameter group of transformations in ε. Derive the infinitesimal forms of the hyperbolic rotation group. Hence by integration rederive the global form of the group.

(b) By making the transformation $\varepsilon = \operatorname{arctanh}(-v/c)$, with v a parameter, show that the hyperbolic rotation group (1) becomes the Lorentz transformation

$$x^* = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \qquad t^* = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}.$$

2. Determine the value of α for which the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^3\mathrm{e}^{2x} - y + \frac{4\mathrm{e}^{-2x}}{y}, \tag{2}$$

is invariant under the one-parameter group

$$x^* = x + \varepsilon, \qquad y^* = y\exp(\alpha\varepsilon).$$

Find the associated invariant and hence find the solution of equation (2).

3. The infinitesimals for the dispersive water-wave equation

$$u_{tt} + 2u_t u_{xx} + 4u_x u_{xt} + 6u_x^2 u_{xx} + u_{xxxx} = 0, \tag{3}$$

are

$$\xi = \alpha x + 2\beta t + \gamma, \qquad \tau = 2\alpha t + \delta, \qquad \phi = \beta x + \kappa,$$

where α, β, γ, δ and κ are arbitrary constants. Determine all the symmetry reductions and find the resulting ordinary differential equations. You do **not** have to solve these ordinary differential equations.

4. Show that the nonlinear equation

$$u_{tt} + u_{xx} + uu_{xx} + u_x^2 + u_{xxtt} = 0, \tag{4}$$

possesses the symmetry reduction

$$u(x,t) = t^{-2}w(z) - 1, \qquad z = x - \lambda\ln t,$$

with λ a constant, and find the ordinary differential equation which $w(z)$ satisfies. Determine whether this is a classical or nonclassical reduction.