2
Basic concepts

2.1 The equations of fluid mechanics

We start with a brief introduction to the equations of fluid mechanics. For further details see for example Batchelor [8] or Acheson [1].

All the fluids considered in this book are assumed to be inviscid and to have constant density ρ (i.e. to be incompressible).

Conservation of momentum yields the Euler equations

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{X}, \tag{2.1}$$

where $\mathbf{u}$ is the vector velocity, p is the pressure and $\mathbf{X}$ is the body force. Here

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \tag{2.2}$$

is the material derivative. We assume that the body force $\mathbf{X}$ derives from a potential Ω, i.e. that

$$\mathbf{X} = -\nabla\Omega. \tag{2.3}$$

In most applications considered in this book, the flow is assumed to be irrotational. Therefore

$$\nabla \times \mathbf{u} = 0. \tag{2.4}$$

Relation (2.4) implies that we can introduce a potential function ϕ such that

$$\mathbf{u} = \nabla\phi. \tag{2.5}$$

Conservation of mass gives

$$\nabla \cdot \mathbf{u} = 0. \tag{2.6}$$

Then (2.5) and (2.6) imply that ϕ satisfies Laplace's equation

$$\nabla^2\phi = 0. \tag{2.7}$$

Flows that satisfy (2.4)–(2.7) are referred to as potential flows. Using the identity

$$\mathbf{u}\cdot\nabla\mathbf{u} = \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) + (\nabla\times\mathbf{u})\times\mathbf{u}, \tag{2.8}$$

(2.4) and (2.2) yield

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial\mathbf{u}}{\partial t} + \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}). \tag{2.9}$$

Substituting (2.9) into (2.1) and using (2.3) and (2.5) we obtain

$$\nabla\left(\frac{\partial\phi}{\partial t} + \frac{\mathbf{u}\cdot\mathbf{u}}{2} + \frac{p}{\rho} + \Omega\right) = 0. \tag{2.10}$$

After integration, (2.10) gives the well-known Bernoulli equation

$$\frac{\partial\phi}{\partial t} + \frac{\mathbf{u}\cdot\mathbf{u}}{2} + \frac{p}{\rho} + \Omega = F(t). \tag{2.11}$$

Here $F(t)$ is an arbitrary function of t. It can be absorbed in the definition of ϕ, and then (2.11) can be rewritten as

$$\frac{\partial\phi}{\partial t} + \frac{\mathbf{u}\cdot\mathbf{u}}{2} + \frac{p}{\rho} + \Omega = B, \tag{2.12}$$

where B is a constant. For steady flows (2.12) reduces to

$$\frac{\mathbf{u}\cdot\mathbf{u}}{2} + \frac{p}{\rho} + \Omega = B. \tag{2.13}$$

2.2 Free surface flows

We introduce the concept of a free surface by contrasting the flow past a sphere (see Figure 2.1) with that of the flow past a bubble (see Figure 2.2). Both flows are assumed to be steady and to approach a uniform stream with a constant velocity U as $x^2 + y^2 + z^2 \to \infty$; the effects of gravity are neglected. They can interpreted as the flows due to a sphere or a bubble rising at a constant velocity U, when viewed in a frame of reference moving with the sphere or the bubble. The pressure p_{b} in the bubble is constant. We denote by S the surface of the sphere or bubble and by $\mathbf{n}$ the outward unit normal.

The flow past a sphere can be formulated as follows:

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad \text{outside} \quad S, \tag{2.14}$$

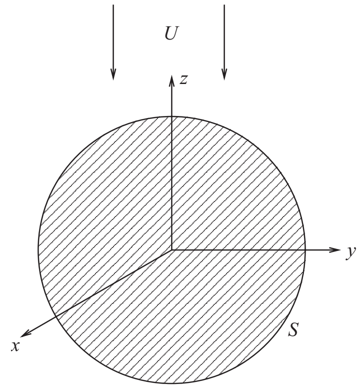

Fig. 2.1. The flow past a rigid sphere. The surface S of the sphere is described by $x^2 + y^2 + z^2 = R^2$, where R is the radius of the sphere.

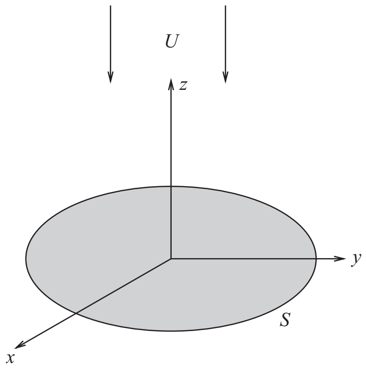

Fig. 2.2. The flow past a bubble. The surface S of the bubble is not known a priori and has to be found as part of the solution.

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad S \tag{2.15}$$

$$(\phi_x, \phi_y, \phi_z) \to (0, 0, -U) \quad \text{as} \quad x^2 + y^2 + z^2 \to \infty. \tag{2.16}$$

Equation (2.14) is Laplace's equation (2.7) expressed in cartesian coordinates. The boundary condition (2.15) is known as the kinematic boundary condition. It states that the normal component of the velocity vanishes on S.

Equations (2.14)–(2.16) form a linear boundary value problem whose solution is

$$\phi = -U\left[z + \frac{R^3 z}{2(x^2+y^2+z^2)^{3/2}}\right]. \tag{2.17}$$

Here R is the radius of the sphere.

We note that we have derived the solution (2.17) without using the Bernoulli equation (2.13), which for the present problem can be written as

$$\frac{1}{2}(\phi_x^2+\phi_y^2+\phi_z^2)+\frac{p}{\rho}=\frac{1}{2}U^2+\frac{p_\infty}{\rho}. \tag{2.18}$$

Here p_∞ denotes the pressure as $x^2+y^2+z^2 \to \infty$. Equation (2.18) holds everywhere outside the sphere. In deriving (2.18) we have set $\Omega = 0$ in (2.13) and evaluated B by taking the limit $x^2+y^2+z^2 \to \infty$ in (2.13). Then, using (2.16) gives $B = U^2/2 + p_\infty/\rho$.

Equation (2.18) is nonlinear but it is only used if we want to calculate the pressure p inside the fluid. In other words the main problem is to find ϕ by solving the linear set of relations (2.14)–(2.16). We may then substitute the values (2.17) of ϕ into the nonlinear equation (2.18) if we wish to compute the pressure.

We now show that we need to use the nonlinear boundary condition (2.18) to solve for the potential ϕ for a flow past the bubble of Figure 2.2. This implies that, because of its nonlinearity, the flow past a bubble is a much harder problem to solve than the flow past a sphere. The potential function ϕ still satisfies (2.14)–(2.16). However, the main difference is that the shape of the surface S of the bubble is not known and has to be found as part of the solution. In other words the equation of the surface S is no longer given as it was for the flow past a sphere. Therefore we need an extra equation to find S. This equation uses (2.18) and can be derived as follows. First we relate the pressure p on the fluid side of S to the pressure p_{b} inside the bubble by using the concept of surface tension. If we draw a line on a fluid surface (such as S), the fluid on the right of the line is found to exert a tension T, per unit length of the line, on the fluid to the left. We call T the surface tension coefficient. It depends on the fluid and also on the temperature. It can be shown (see for example Batchelor [8]) that

$$p - p_{\mathrm{b}} = TK = T\left(\frac{1}{R_1}+\frac{1}{R_2}\right). \tag{2.19}$$

Here R_1 and R_2 are the principal radii of curvature of the fluid surface: they are counted positive when the centres of curvature lie inside the fluid. The quantity

$$K = \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \tag{2.20}$$

is referred to as the mean curvature of the fluid surface. In most applications presented in this book the surface tension T is assumed to be constant.

We now apply the Bernoulli equation (2.18) to the fluid side of the surface S and use (2.19). This gives

$$\frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + \frac{T}{\rho}K = \frac{1}{2}U^2 + \frac{p_\infty - p_\mathrm{b}}{\rho} \quad \text{on} \quad S. \tag{2.21}$$

Equation (2.21) is known as the dynamic boundary condition. This is the extra equation needed to find S. To solve the bubble problem we seek the function ϕ and the equation of the surface S such that (2.14)–(2.16) and (2.21) are satisfied. It is a nonlinear problem that requires the solution of a partial differential equation (here the Laplace equation (2.14)) in a domain whose boundary (here S) has to be found as part of the solution. This is a typical free surface flow problem. In this book we will describe various analytical and numerical methods for investigating such nonlinear problems.

We note that the problem of Figure 2.2 is an idealised one, in which the viscosity and gravity and the wake behind the bubble are neglected. Bubbles with wakes and the effect of including gravity will be considered in Section 3.4.3. Readers interested in the effects of viscosity are referred to, for example, [117].

The dynamic boundary condition (2.21) is valid for steady flows with $\Omega = 0$. Combining (2.12) and (2.19) we find that the general form of the dynamic boundary condition (for unsteady flows) with $\Omega \neq 0$ is

$$\frac{\partial \phi}{\partial t} + \frac{\mathbf{u} \cdot \mathbf{u}}{2} + \Omega + \frac{T}{\rho}K = B. \tag{2.22}$$

Here B is the Bernoulli constant. For steady flows, (2.22) reduces to

$$\frac{\mathbf{u} \cdot \mathbf{u}}{2} + \Omega + \frac{T}{\rho}K = B. \tag{2.23}$$

2.3 Two-dimensional flows

As we shall see, many interesting free surface flows can be modelled as two-dimensional flows. We then introduce cartesian coordinates x and y with the y-axis directed vertically upwards (at present we reserve the letter z to denote the complex quantity $x + iy$). In most applications considered in this book, the potential Ω (see (2.3)) is due to gravity. Assuming that the acceleration of gravity g is acting in the negative y-direction, we write Ω as

$$\Omega = gy. \tag{2.24}$$

An example is the two-dimensional free surface flow past a semicircular obstacle at the bottom of a channel (see Figure 2.3). This two-dimensional configuration provides a good approximation to the three-dimensional free surface flow past a long half-cylinder perpendicular to the plane of the figure (except near the ends of the cylinder). The cross section of the cylinder is the semicircle shown in Figure 2.3.

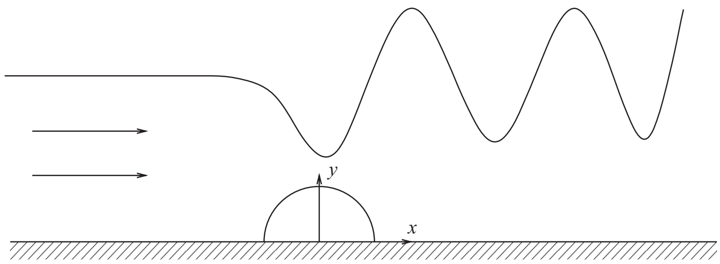

Fig. 2.3. Sketch of two-dimensional free surface flow past a submerged semicircle.

For two-dimensional potential flows, (2.4) and (2.6) become

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0, \tag{2.25}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2.26}$$

Here u and v are the x- and y- components of the velocity vector $\mathbf{u}$.

We can introduce a streamfunction ψ by noting that (2.26) is satisfied by

$$u = \frac{\partial \psi}{\partial y}, \tag{2.27}$$

$$v = -\frac{\partial \psi}{\partial x}. \tag{2.28}$$

It then follows from (2.25) that

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \tag{2.29}$$

For two-dimensional flows, equations (2.5) and (2.7) give

$$u = \frac{\partial \phi}{\partial x}, \tag{2.30}$$

$$v = \frac{\partial \phi}{\partial y} \tag{2.31}$$

and

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{2.32}$$

Combining (2.27), (2.28), (2.30) and (2.31) we obtain

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \tag{2.33}$$

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{2.34}$$

Equations (2.33) and (2.34) can be recognised as the classical Cauchy–Riemann equations. They imply that the complex potential

$$f = \phi + i\psi \tag{2.35}$$

is an analytic function of $z = x + iy$ in the flow domain. This result is particularly important since it implies that two-dimensional potential flows can be investigated by using the theory of analytic functions. This applies in particular to all two-dimensional potential free surface flows with or without gravity and/or surface tension included in the dynamic boundary condition. It does not apply, however, to axisymmetric and three-dimensional free surface flows. Since the derivative of an analytic function is also an analytic function, it follows that the complex velocity

$$u - iv = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial y} + i\frac{\partial \psi}{\partial x} = \frac{df}{dz} \tag{2.36}$$

is also an analytic function of $z = x + iy$.

The theory of analytic functions will be used intensively in the following chapters to study two-dimensional free surface flows. In particular the following important tools will be useful.

The first tool is conformal mappings. These are changes of variable defined by analytic functions. For example, if $h(t)$ is an analytic function of t, the change of variables $z = h(t)$ enables us to seek the complex velocity $u - iv$ as an analytic function of t (since, as mentioned above, an analytic function of an analytic function is also an analytic function). Such conformal mappings are used to redefine a problem in a new complex t-plane in which the geometry is simpler than in the original z-plane.

The second tool is Cauchy's theorem: *If $h(z)$ is analytic throughout a simply connected domain D then, for every closed contour C within D,*

$$\int_C h(z)dz = 0. \tag{2.37}$$

The third tool is the Cauchy integral formula: *Let $h(z)$ be analytic everywhere within and on a closed contour C, taken in the positive sense (counterclockwise). Then the integral*

$$\frac{1}{2i\pi}\int_C \frac{h(z)}{z-z_0}dz \tag{2.38}$$

takes the following values:

$$0 \quad \text{if} \quad z_0 \quad \text{is outside } C, \tag{2.39}$$

$$h(z_0) \quad \text{if} \quad z_0 \quad \text{is inside } C, \tag{2.40}$$

$$\frac{1}{2}h(z_0) \quad \text{if} \quad z_0 \quad \text{is on } C. \tag{2.41}$$

When z_0 is on C the integral (2.38) is a Cauchy principal value.

We now show that for steady flows the streamfunction ψ is constant along streamlines. A streamline is a line to which the velocity vectors are tangent. Let us describe a streamline in parametric form by $x = X(s)$, $y = Y(s)$, where s is the arc length. Then we have

$$-vX'(s) + uY'(s) = 0, \tag{2.42}$$

where the primes denote derivatives with respect to s. Using (2.27) and (2.28) we have

$$\frac{\partial\psi}{\partial x}X'(s) + \frac{\partial\psi}{\partial y}Y'(s) = \frac{d\psi}{ds} = 0, \tag{2.43}$$

which implies that ψ is constant along a streamline. For steady flows the kinematic boundary condition implies that a free surface is a streamline. The streamfunction is therefore constant along a free surface.

For two-dimensional flows the dynamic boundary condition (2.22) becomes

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\phi_x^2 + \phi_y^2) + gy + \frac{T}{\rho}K = B. \tag{2.44}$$

If we denote by θ the angle between the tangent to the free surface and the horizontal then the curvature K can be defined by

$$K = -\frac{d\theta}{ds} \tag{2.45}$$

where again s denotes the arc length. In particular if the (unknown) equation of the free surface is $y = \eta(x,t)$ then

$$\tan\theta = \eta_x \quad \text{and} \quad \frac{dx}{ds} = \frac{1}{(1+\eta_x^2)^{1/2}}. \tag{2.46}$$

Using (2.45), (2.46) and the chain rule gives the formula

$$K = -\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}. \tag{2.47}$$

2.4 Linear waves

2.4.1 The water-wave equations

Many free surface flows involve waves on their free surfaces. When dissipation is neglected and the flow is assumed to be two-dimensional, these waves approach uniform wave trains in the far field (see for example Figure 2.3). Therefore a fundamental problem in the theory of free surface flows is the study of a uniform train of two-dimensional waves of wavelength λ extending from $x = -\infty$ to $x = \infty$ and travelling at a constant velocity c. The flow configuration is illustrated in Figure 2.4.

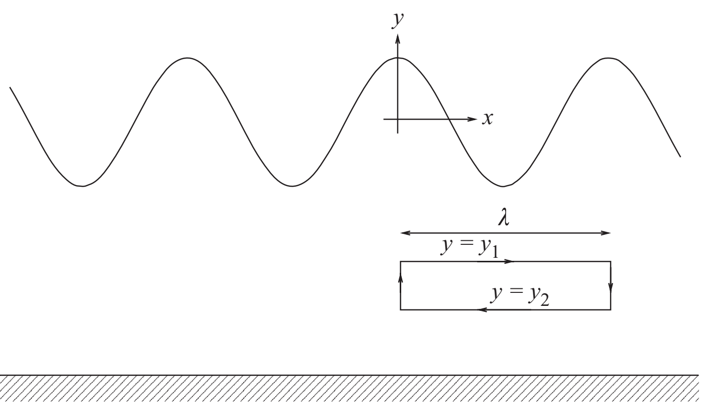

Fig. 2.4. A two-dimensional train of waves viewed in a frame of reference moving with the wave. The free surface profile has wavelength λ. The fluid is bounded below by a horizontal bottom with equation $y = -h$. Also shown is the rectangular contour used in (2.56).

Here for convenience we have chosen a frame of reference moving with the wave, so that the flow is steady. Using the notation of Section 2.3, we formulate the problem as

$$\phi_{xx} + \phi_{yy} = 0, \quad -h < y < \eta(x), \tag{2.48}$$

$$\phi_y = \phi_x \eta_x \quad \text{on} \quad y = \eta(x), \tag{2.49}$$

$$\phi_y = 0 \quad \text{on} \quad y = -h, \tag{2.50}$$

$$\frac{1}{2}(\phi_x^2 + \phi_y^2) + gy - \frac{T}{\rho}\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} = B \quad \text{on} \quad y = \eta(x), \tag{2.51}$$

$$\nabla\phi(x+\lambda, y) = \nabla\phi(x, y), \tag{2.52}$$

$$\eta(x+\lambda) = \eta(x), \tag{2.53}$$

$$\int_0^\lambda \eta(x)dx = 0, \tag{2.54}$$

$$\frac{1}{\lambda}\int_0^\lambda \phi_x dx = c \quad \text{on} \quad y = \text{constant}. \tag{2.55}$$

Here g is the acceleration of gravity (assumed to act in the negative y-direction), T is the surface tension, ρ is the density, $y = -h$ is the equation of the bottom and $y = \eta(x)$ is the equation of the (unknown) free surface. Equations (2.49) and (2.50) are the kinematic boundary conditions on the free surface and on the bottom respectively. Equation (2.51) is the dynamic boundary condition on the free surface. We have used (2.44) and the formula (2.47) for the curvature of a curve $y = \eta(x)$. Relations (2.52) and (2.53) are periodicity conditions, which require the solution to be periodic with wavelength λ. Equation (2.54) fixes the origin of the y-coordinates as the mean water level. Finally, (2.55) defines the velocity c as the average value of $u = \phi_x$ at a level $y =$ constant in the fluid. The value of c is independent of the constant chosen; this can be seen by applying Stokes' theorem to the vector velocity (u, v) using a contour C consisting of two horizontal lines $y = y_1$, $y = y_2$ and two vertical lines separated by a wavelength (see Figure 2.4). Since the flow is irrotational,

$$\int_C u dx + v dy = 0. \tag{2.56}$$

The contributions from the two vertical lines cancel by periodicity and (2.56) gives

$$\int_0^\lambda [u]_{y=y_1}\,dx = \int_0^\lambda [u]_{y=y_2}\,dx. \tag{2.57}$$

Since y_1 and y_2 are arbitrary, the integral on the left-hand side of (2.55) is independent of the level $y =$ constant chosen in the fluid.

The relations (2.48)–(2.55) are referred to as the water-wave equations because they model waves travelling at the interface between water and air (although they apply also to other fluids).

2.4.2 Linear solutions for water waves

A trivial solution of the system (2.48)–(2.55) is

$$\phi = cx, \quad \eta(x) = 0 \quad \text{and} \quad B = \frac{c^2}{2}. \tag{2.58}$$

This solution describes a uniform stream with constant velocity c, bounded below by a horizontal bottom and above by a flat free surface.

Linear waves are obtained by seeking a solution as a small perturbation of the exact solution (2.58). Therefore we write

$$\phi(x, y) = cx + \varphi(x, y) \tag{2.59}$$

and assume that both $|\varphi(x, y)|$ and $|\eta(x)|$ are small. Substituting (2.59) into (2.48)–(2.55) and dropping nonlinear terms in φ and η, we obtain the linear system

$$\varphi_{xx} + \varphi_{yy} = 0, \quad -h < y < 0, \tag{2.60}$$

$$\varphi_y = c\eta_x, \quad y = 0, \tag{2.61}$$

$$\varphi_y = 0, \quad y = -h, \tag{2.62}$$

$$-\frac{T}{\rho}\eta_{xx} + c\varphi_x + g\eta = 0, \quad y = 0, \tag{2.63}$$

$$\nabla\varphi(x + \lambda, y) = \nabla\varphi(x, y), \tag{2.64}$$

$$\eta(x + \lambda) = \eta(x), \tag{2.65}$$

$$\int_0^\lambda \eta(x)dx = 0, \tag{2.66}$$

$$\frac{1}{\lambda}\int_0^{\lambda} \varphi_x dx = 0 \quad \text{on} \quad y = \text{constant}. \tag{2.67}$$

We choose the origin of x at a crest and assume that the wave is symmetric about $x = 0$. Thus we impose the conditions

$$\varphi(-x, y) = -\varphi(x, y), \tag{2.68}$$

$$\eta(-x) = \eta(x). \tag{2.69}$$

Using the method of separation of variables, we seek a solution of (2.60) in the form

$$\varphi(x, y) = X(x)Y(y). \tag{2.70}$$

Substituting (2.70) into (2.60), (2.68) and (2.62) yields

$$X(-x) = -X(x), \tag{2.71}$$

the ordinary differential equations

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \text{constant} = -\alpha^2, \tag{2.72}$$

and the boundary condition

$$Y'(-h) = 0. \tag{2.73}$$

Here we have chosen a negative separation constant in (2.72), so that the solution is periodic in x. Solutions of the two ordinary differential equations (2.72) satisfying (2.71) and (2.73) are written as

$$X(x) = \sin \alpha x, \tag{2.74}$$

$$Y(y) = \cosh \alpha(y + h). \tag{2.75}$$

The periodicity condition (2.64) implies that

$$\alpha = nk, \tag{2.76}$$

where n is a positive integer and

$$k = \frac{2\pi}{\lambda} \tag{2.77}$$

is the wavenumber. Multiplying (2.74) and (2.75) and taking a linear combination of the solutions corresponding to the values (2.76) of α, we obtain

$$\varphi(x, y) = \sum_{n=1}^{\infty} B_n \cosh nk(y + h) \sin nkx. \tag{2.78}$$

Here the B_n are constants.

Using the periodicity and the symmetry conditions (2.69) and (2.65), we express $\eta(x)$ as the Fourier series

$$\eta(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos nkx \tag{2.79}$$

where the A_n are constants. The condition (2.66) implies that $A_0 = 0$.

Substituting (2.78) and (2.79) into (2.61) and equating the coefficients of $\sin nkx$ yields

$$cA_n = -B_n \sinh nkh, \quad n = 1, 2, \ldots \tag{2.80}$$

Similarly substituting (2.78) and (2.79) into (2.63) gives

$$\frac{T}{\rho} A_n n^2 k^2 + gA_n + cB_n nk \cosh nkh = 0, \quad n = 1, 2, \ldots \tag{2.81}$$

Eliminating B_n between (2.80) and (2.81) yields

$$\left(g + \frac{T}{\rho} n^2 k^2 - \frac{c^2 nk}{\sinh nkh} \cosh nkh \right) A_n = 0, \quad n = 1, 2, \ldots \tag{2.82}$$

Since we seek a nontrivial periodic solution $\eta(x) \neq 0$, we can assume without loss of generality that $A_1 \neq 0$; then (2.82) with $n = 1$ implies that

$$c^2 = \left(\frac{g}{k} + \frac{T}{\rho} k \right) \tanh kh. \tag{2.83}$$

Relation (2.82) for $n > 1$ gives

$$A_n = 0, \quad n = 2, 3, \ldots, \tag{2.84}$$

provided that

$$g + \frac{T}{\rho} n^2 k^2 - \frac{c^2 nk}{\sinh nkh} \cosh nkh \neq 0, \quad n = 2, 3, \ldots \tag{2.85}$$

If (2.85) is satisfied, the solution of the linear problem is

$$\varphi = -\frac{cA_1}{\sinh kh} \cosh k(y + h) \sin kx, \tag{2.86}$$

$$\eta = A_1 \cos kx. \tag{2.87}$$

If the condition (2.85) is not satisfied for some integer value m of n, the solution of the linear problem is

$$\varphi = -\frac{cA_1}{\sinh kh} \cosh k(y + h) \sin kx - \frac{cA_m}{\sinh mkh,} \cosh mk(y + h) \sin mkx, \tag{2.88}$$

$$\eta_1 = A_1 \cos kx + A_m \cos mkx, \tag{2.89}$$

where A_m is an arbitrary constant. In the theory of linear waves, it is usually assumed that $A_m = 0$. However, when we are developing nonlinear theories for water waves in Chapters 5 and 6, i.e. improving the linear approximations (2.88) and (2.89) by adding nonlinear corrections or solving the fully nonlinear problem (2.48)–(2.55) numerically, we shall see that $A_m \neq 0$. Two consequences are the existence of many different families of nonlinear periodic gravity–capillary waves and the existence of solitary waves with oscillatory tails.

The velocity c is called the phase velocity and equation (2.83) is the (linear) dispersion relation. Relation (2.83) implies that waves of different wavenumbers and therefore of different wavelengths travel at different phase velocities c.

It is convenient to rewrite (2.83) in the dimensionless form

$$F^2 = \left(\frac{1}{kh} + \tau kh\right) \tanh kh, \tag{2.90}$$

where

$$F = \frac{c}{(gh)^{1/2}} \tag{2.91}$$

is the Froude number and

$$\tau = \frac{T}{\rho g h^2} \tag{2.92}$$

is the Bond number. Relation (2.90) is shown graphically in Figure 2.5, where we present values of F^2 versus $1/(kh) = \lambda/(2\pi h)$ for four values of τ.

The curves of Figure 2.5 illustrate that F^2 is a monotonically decreasing function of λ/h when $\tau > 1/3$ and that it has a minimum for $\tau < 1/3$. As $\lambda/h \to \infty$, $F \to 1$. The different behaviours for $\tau < 1/3$ (minimum) and $\tau > 1/3$ (monotone decay) in Figure 2.5 have many implications, in particular for the study of nonlinear periodic and solitary gravity–capillary waves (see Chapters 5 and 6).

We now examine two particular cases.

The first is the case of water of infinite depth. This is obtained by taking the limit $kh \to \infty$ in (2.83), (2.86) and (2.87) and leads to

$$\varphi = -cA_1 e^{ky} \sin kx, \tag{2.93}$$

$$\eta = A_1 \cos kx, \tag{2.94}$$

$$c^2 = \frac{g}{k} + \frac{T}{\rho} k. \tag{2.95}$$

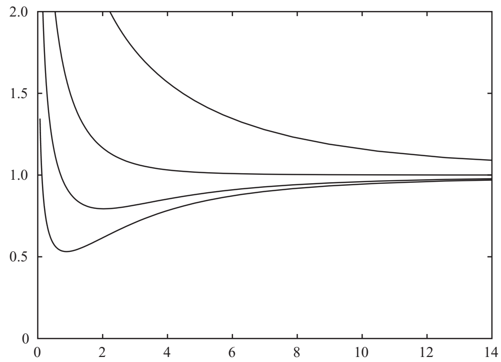

Fig. 2.5. Values of F^2 versus $1/(kh)$. The curves correspond from top to bottom to $\tau = 1.3$, $\tau = 1/3$, $\tau = 0.1$ and $\tau = 0.05$. For $\tau < 1/3$ the curves have a minimum whereas for $\tau > 1/3$ the curves are monotonically decreasing.

Since $kh = 2\pi h/\lambda$, the infinite-depth results (2.93)–(2.95) provide an approximation to the finite-depth results (2.83), (2.86) and (2.87) when the wavelength λ is small compared with the depth h.

Waves with $g \neq 0$, $T = 0$ are referred to as pure gravity waves. They are characterised in the case of infinite depth by the dispersion relation

$$c^2 = \frac{g}{k}. \tag{2.96}$$

Similarly, waves with $g = 0$, $T \neq 0$ are called pure capillary waves and are characterised in the infinite-depth case by the dispersion relation

$$c^2 = \frac{T}{\rho}k. \tag{2.97}$$

A simple calculation based on (2.95) shows that c^2 reaches a minimum value given by

$$c_{\min} = \left(\frac{4Tg}{\rho}\right)^{1/4} \tag{2.98}$$

when

$$k = k_{\min} = \left(\frac{\rho g}{T}\right)^{1/2}. \tag{2.99}$$

Graphs of c versus λ in units of $c_{\min}$ and $\lambda_{\min} = 2\pi/k_{\min}$ are shown in Figure 2.6.

The solid curve corresponds to (2.95), the dotted curve to (2.97), and the broken curve to (2.96). These curves show that waves with $\lambda > \lambda_{\min}$ are dominated by gravity and can be approximated by pure gravity waves for

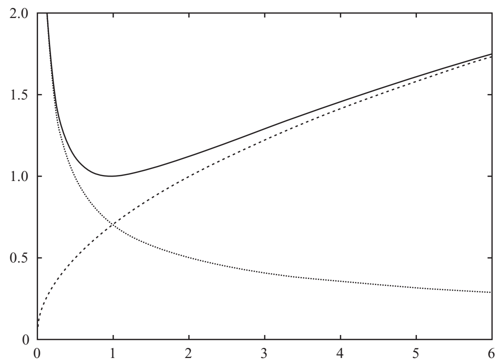

Fig. 2.6. Values of c versus $\lambda = 2\pi/k$ in units of $c_{\min}$ and $\lambda_{\min}$. The solid curve corresponds to (2.95), the dotted curve to (2.97) and the broken curve to (2.96).

λ large. Waves with $\lambda < \lambda_{\min}$ are dominated by surface tension and can be approximated by pure capillary waves for λ small.

The second particular case is that of pure gravity waves (i.e. $T = 0$) in water of finite depth. Then (2.90) reduces to

$$khF^2 = \tanh kh. \tag{2.100}$$

Since

$$\frac{d}{d\,(kh)} \tanh kh \leq 1, \tag{2.101}$$

equation (2.100) has the solution $kh > 0$ when $F < 1$. For $F > 1$ the only real solution of (2.100) is $kh = 0$. This implies that linear gravity waves only exist when $F < 1$; for $F > 1$, linear gravity waves are not possible. Flows characterised by $F < 1$ are called subcritical and those characterised by $F > 1$ are called supercritical. The distinction between subcritical and supercritical flows will appear often in this book.

So far we have discussed linear waves in a frame of reference moving with the wave. This is a convenient choice because the flow is then steady. However, it is also useful to look at waves from the point of view of a fixed frame of reference in which the wave moves to the left at a constant velocity c. The nonlinear governing equations are then

$$\phi_{xx} + \phi_{yy} = 0, \quad -h < y < \eta(x,t), \tag{2.102}$$

$$\eta_t = \phi_y - \phi_x \eta_x \quad \text{on} \quad y = \eta(x,t), \tag{2.103}$$

$$\phi_y = 0 \quad \text{on} \quad y = -h, \tag{2.104}$$

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) + gy - \frac{T}{\rho}\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} = B \quad \text{on} \quad y = \eta(x,t), \tag{2.105}$$

$$\phi(x+\lambda, y, t) = \phi(x, y, t), \tag{2.106}$$

$$\eta(x+\lambda, t) = \eta(x, t), \tag{2.107}$$

$$\int_0^\lambda \eta(x,t)dx = 0. \tag{2.108}$$

A trivial solution of the system (2.102)–(2.108) is

$$\phi = 0, \quad \eta = 0 \quad \text{and} \quad B = 0. \tag{2.109}$$

We can construct linear waves by assuming a small perturbation of the exact solution (2.109) in the form of a wave travelling to the left at a constant velocity c. Therefore we rewrite ϕ and η in terms of two new functions $\bar{\phi}$ and $\bar{\eta}$:

$$\phi(x,y,t) = \bar{\phi}(x+ct, y) \quad \text{and} \quad \eta(x,t) = \bar{\eta}(x+ct) \tag{2.110}$$

Substituting (2.110) into the system (2.102)–(2.108) and dropping nonlinear terms in $\bar{\phi}$ and $\bar{\eta}$, we obtain the linear system

$$\bar{\phi}_{xx} + \bar{\phi}_{yy} = 0, \quad -h < y < 0, \tag{2.111}$$

$$c\bar{\eta}_x = \bar{\phi}_y \quad \text{on} \quad y = 0, \tag{2.112}$$

$$\bar{\phi}_y = 0 \quad \text{on} \quad y = -h, \tag{2.113}$$

$$c\bar{\phi}_x + g\bar{\eta} - \frac{T}{\rho}\bar{\eta}_{xx} = 0 \quad \text{on} \quad y = 0, \tag{2.114}$$

$$\bar{\phi}(x+\lambda+ct, y) = \bar{\phi}(x+ct, y), \tag{2.115}$$

$$\bar{\eta}(x+\lambda+ct) = \bar{\eta}(x+ct), \tag{2.116}$$

$$\int_0^\lambda \bar{\eta}(x+ct)dx = 0. \tag{2.117}$$

Following the derivation of (2.86)–(2.89), we find that the solution of (2.111)–(2.117) is

$$\bar{\phi} = -\frac{cA_1}{\sinh kh}\cosh k(y+h)\sin k(x+ct), \tag{2.118}$$

$$\bar{\eta} = A_1 \cos k(x+ct) \tag{2.119}$$

if (2.85) is satisfied and

$$\bar{\phi} = -\frac{cA_1}{\sinh kh}\cosh k(y+h)\sin k(x+ct),$$

$$-\frac{cA_m}{\sinh mkh}\cosh mk(y+h)\sin mk(x+ct), \qquad (2.120)$$

$$\bar{\eta} = A_1 \cos k(x+ct) + A_m \cos mk(x+ct) \qquad (2.121)$$

if for $n = m$ (2.85) is not satisfied. The dispersion relation is given as before by (2.83).

2.4.3 Superposition of linear waves

Since the system (2.111)–(2.117) is linear, new solutions can be obtained by superposing solutions corresponding to different values of k and/or of A_1.

We consider two particular superpositions for the solution (2.118), (2.119).

The first corresponds to the superposition of two waves of the same amplitude travelling at the same velocity but in opposite directions. This gives

$$\eta = A_1 \cos k(x+ct) + A_1 \cos k(x-ct), \qquad (2.122)$$

$$\phi = -\frac{cA_1}{\sinh kh}\cosh k(y+h)\sin k(x+ct)$$
$$+\frac{cA_1}{\sinh kh}\cosh k(y+h)\sin k(x-ct). \qquad (2.123)$$

Using the trigonometric identities

$$\cos p + \cos q = 2\cos\frac{p+q}{2}\cos\frac{p-q}{2} \qquad (2.124)$$

$$\sin p + \sin q = 2\sin\frac{p+q}{2}\cos\frac{p-q}{2} \qquad (2.125)$$

we can rewrite (2.122), (2.123) as

$$\eta = 2A_1 \cos kx \cos kct, \qquad (2.126)$$

$$\phi = -2\frac{cA_1}{\sinh kh}\cosh k(y+h)\cos kx \sin kct. \qquad (2.127)$$

The solution defined by (2.126), (2.127) is known as a linear standing wave because the position of its nodal points and of the maximum displacement of the free surface are fixed as t varies. The wave does not propagate and its free surface moves periodically up and down as t varies. The period of this motion is

$$T_{\mathrm{s}} = \frac{2\pi}{kc}. \tag{2.128}$$

Since $u = \phi_x = 0$ along the lines $x = 0$ and $x = \pi/k = \lambda/2$, we can replace these two lines by walls (the kinematic boundary condition on them is then automatically satisfied). The resulting flow models the periodic sloshing of a liquid in a container (see Figure 2.7).

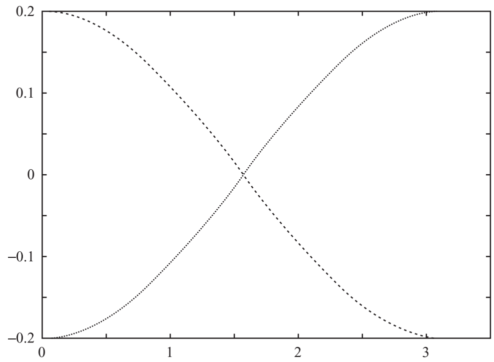

Fig. 2.7. Standing wave for $A_1 = 0.1$ and $k = 1$. The broken line is the free surface profile at $t = 0$ and the dotted line is the free surface profile at $t = T_{\mathrm{s}}/2$. The flow models sloshing in a container bounded by two vertical walls at $x = 0$ and $x = \pi$.

An interesting question is whether there are similar nonlinear solutions. This question is addressed in Chapter 11, where we construct analytical approximations to such solutions.

The second example of superposition that we consider is that of two wave trains of the same amplitude travelling in the same direction but with slightly different wavenumbers k and $\bar{k}$. We first introduce the angular frequency

$$\omega = kc \tag{2.129}$$

and write $\omega = W(k)$. Using (2.83) we have

$$W(k) = k\left[\left(\frac{g}{k} + \frac{T}{\rho}k\right)\tanh kh\right]^{1/2}. \tag{2.130}$$

Next we rewrite (2.118) and (2.119) as

$$\bar{\eta} = A_1 \cos[kx + W(k)t], \tag{2.131}$$

$$\bar{\phi} = -\frac{cA_1}{\sinh kh}\cosh k(y+h)\sin[kx + W(k)t]. \tag{2.132}$$

The superposition described above then yields

$$\bar{\eta} = A_1 \cos[kx + W(k)t] + A_1 \cos[\bar{k}x + W(\bar{k})t]. \qquad (2.133)$$

Using the identity (2.124), we can rewrite (2.133) as

$$\bar{\eta} = 2A_1 \cos\left\{\left[\frac{1}{2}[(k+\bar{k})]x + \frac{1}{2}[W(k)+W(\bar{k}))t\right]\right\}$$

$$\times \cos\left\{\frac{1}{2}[(k-\bar{k})]x + \frac{1}{2}[(W(k)-W(\bar{k})t]\right\}. \qquad (2.134)$$

For $\bar{k}$ close to k, we may approximate (2.134) by

$$\bar{\eta} = \alpha(x,t)\cos[kx + W(k)t], \qquad (2.135)$$

where

$$\alpha(x,t) = 2A_1 \cos\left\{\frac{1}{2}(k-\bar{k})x + \frac{1}{2}[W(k)-W(\bar{k})]t\right\}. \qquad (2.136)$$

The expression (2.135) is the same as (2.131) except that the constant amplitude A_1 has been replaced by the variable amplitude $\alpha(x,t)$.

Differentiating (2.136) with respect to x and t yields

$$\frac{\partial \alpha}{\partial x} = -A_1(k-\bar{k})\sin\left\{\frac{1}{2}(k-\bar{k})x + \frac{1}{2}[W(k)-W(\bar{k})]t\right\} \qquad (2.137)$$

and

$$\frac{\partial \alpha}{\partial t} = -A_1[W(k)-W(\bar{k})]\sin\left\{\frac{1}{2}(k-\bar{k})x + \frac{1}{2}[W(k)-W(\bar{k})]t\right\}. \qquad (2.138)$$

The derivatives (2.137) and (2.138) are of order $k - \bar{k}$ and $W(k) - W(\bar{k})$ respectively. They are therefore small for $\bar{k}$ close to k. This implies that the amplitude $\alpha(x,t)$ is a slowly varying function of x and t. In other words, the solution is a wave of wavenumber k, travelling at velocity c, whose amplitude $\alpha(x,t)$ is slowly modulated. The amplitude $\alpha(x,t)$ is itself a wave travelling at velocity

$$\frac{W(k)-W(\bar{k})}{k-\bar{k}}. \qquad (2.139)$$

For $\bar{k}$ close to k, the velocity (2.139) becomes

$$c_{\mathrm{g}} = \frac{dW(k)}{dk}. \qquad (2.140)$$

The velocity c_{g} is called the group velocity. In general it differs from the phase velocity

$$c = \frac{W(k)}{k}. \tag{2.141}$$

For water waves, (2.130) and (2.140) give

$$c_{\mathrm{g}} = \frac{1}{2}\left[\left(gk + \frac{T}{\rho}k^3\right)\tanh kh\right]^{-1/2} \times \left[\left(g + \frac{3Tk^2}{\rho}\right)\tanh kh + \left(g + \frac{T}{\rho}k^2\right)\frac{kh}{\cosh^2 kh}\right]. \tag{2.142}$$

We now examine in more detail the case of infinite depth. The phase velocity c is then given by (2.95). Taking the limit $kh \to \infty$ in (2.142), we obtain

$$c_{\mathrm{g}} = \frac{1}{2}\left(gk + \frac{Tk^3}{\rho}\right)^{-1/2}\left(g + \frac{3Tk^2}{\rho}\right). \tag{2.143}$$

In particular, we have for pure gravity waves ($g \neq 0$, $T = 0$)

$$c_{\mathrm{g}} = \frac{1}{2}\left(\frac{g}{k}\right)^{1/2} = \frac{c}{2} \tag{2.144}$$

and for pure capillary waves ($T \neq 0$, $g = 0$)

$$c_{\mathrm{g}} = \frac{3}{2}\left(\frac{Tk}{\rho}\right)^{1/2} = \frac{3c}{2}. \tag{2.145}$$

The phase velocity c is the velocity at which the wave travels. The group velocity c_{g} is the velocity at which the slowly varying amplitude travels. This phenomenon is illustrated in Figure 2.8, where we present the solution (2.135) for pure gravity waves of infinite depth.

Here we have assumed $g = 1$, $k = 1$, $\bar{k} = 1.1$ and $A_1 = 0.2$ and have chosen $t = 0$. The outside curves are the envelope of the wave train. Both the wave train and the envelope travel to the left. Using (2.96) we find that the wave train travels at the speed $c = 1$ whereas (2.144) shows that the envelope travels at the speed $c_{\mathrm{g}} = 1/2$. Since $c_{\mathrm{g}} < c$ the waves will advance in their envelope, and as they approach the nodal points of their envelope they will progressively die out. However, waves are born just ahead of the nodal points of the envelope. These graphical results illustrate that the wave travels at the velocity c whereas the envelope of the wave (i.e. the amplitude) travels at the velocity c_{g}.

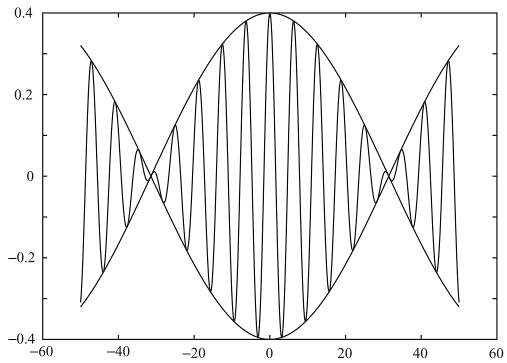

Fig. 2.8. The solution (2.135) for $A_1 = 0.2$, $k = 1$ and $\bar{k} = 1.1$.

A simple relation between c and $c_{\rm g}$ can be derived by combining (2.129) and (2.140) to give

$$c_{\rm g} = c + k\frac{dc}{dk}. \tag{2.146}$$

Relation (2.146) shows that if c has a minimum for some value of k, then $c = c_{\rm g}$ at this minimum (since $dc/dk = 0$ at a minimum). For example, in water of infinite depth c has a minimum for $k = k_{\rm min}$, where $k_{\rm min}$ is given by (2.99) (see also Figure 2.6), and $c_{\rm g} = c$ when $k = k_{\rm min}$. On the right of the minimum in Figure 2.6 we have $dc/dk < 0$, and (2.146) implies that $c_{\rm g} < c$. Similarly, $dc/dk > 0$ on the left of the minimum in Figure 2.6 and $c_{\rm g} > c$.

One important property of the group velocity is that it is the speed at which the energy of a linear wave travels. We will demonstrate this property in the particular case of pure gravity waves in water of finite depth. The analysis is similar to that presented in Billingham and King [13]. At a fixed value of x, the rate at which the fluid on the left does work on the fluid on the right is given by

$$\int_{-h}^{0} p\frac{\partial \phi}{\partial x}dy. \tag{2.147}$$

The average of (2.147) over one period is

$$E_f = \frac{\omega}{2\pi}\int_{t^*}^{t^*+2\pi/\omega}\int_{-h}^{0} p\frac{\partial \phi}{\partial x}dydt, \tag{2.148}$$

where t^* is an arbitrary value of t and ω is the angular frequency. The value of p is obtained by linearising (2.12) (with $\Omega = gy$) around $\mathbf{u} = \mathbf{0}$. This gives

$$p = -\rho\frac{\partial\phi}{\partial t} - \rho g y + \text{ constant.} \tag{2.149}$$

Using (2.118) we find

$$\int_{t^*}^{t^*+2\pi/\omega} \frac{\partial\phi}{\partial x}dt = -\frac{cA_1k}{\sinh kh}\cosh k(y+h)\int_{t^*}^{t^*+2\pi/\omega}\cos k(x+ct)dt = 0. \tag{2.150}$$

Therefore (2.148) simplifies to

$$E_f = -\rho\frac{\omega}{2\pi}\int_{t^*}^{t^*+2\pi/\omega}\int_{-h}^{0}\frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial x}dydt. \tag{2.151}$$

Substituting (2.118) into (2.151) and evaluating the integral yields

$$E_f = \frac{\rho A_1^2k^2c^3}{4\,\sinh^2 kh}\left(h + \frac{\sinh 2kh}{2k}\right). \tag{2.152}$$

We now define the kinetic and potential energy per unit horizontal length by

$$\int_{-h}^{0}\frac{1}{2}\rho\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]dy \tag{2.153}$$

and

$$\int_0^{\eta}\rho g y dy = \frac{1}{2}\rho g\eta^2. \tag{2.154}$$

Averaging the quantities (2.153) and (2.154) over a wavelength gives the mean kinetic energy

$$\bar{K} = \frac{\rho}{2\lambda}\int_0^{\lambda}\int_{-h}^{0}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]dydx \tag{2.155}$$

and the mean potential energy

$$\bar{V} = \frac{\rho g}{2\lambda}\int_0^{\lambda}\eta^2 dx. \tag{2.156}$$

Substituting (2.118) and (2.119) into (2.155) and (2.156) gives, after integration,

$$\bar{K} = \bar{V} = \frac{1}{4}\rho g A_1^2. \tag{2.157}$$

Thus the total energy is

$$E = \bar{K} + \bar{V} = \frac{1}{2}\rho g A_1^2. \tag{2.158}$$

Combining (2.152) and (2.158) we obtain

$$E_f = E\frac{1}{2}c\left(1 + \frac{2kh}{\sinh 2kh}\right). \tag{2.159}$$

Using (2.142) with $T = 0$ gives

$$c_{\mathrm{g}} = \frac{c}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right). \tag{2.160}$$

Therefore comparing (2.159) and (2.160) yields

$$E_f = Ec_{\mathrm{g}}. \tag{2.161}$$

This shows that the energy in the wave travels at the group velocity c_{g}. This property will be used in Chapter 4, where we discuss the radiation condition.